PRACTICE MIDTERM 1 – SOLUTIONS

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1. (a) Let f be a function and a, L real numbers. Define carefully: $\lim_{x\to a} f(x) = L$ if and only if...

For every number $\epsilon > 0$ there is a number $\delta > 0$ such that for all x in $\mathbb{R}$,

if $\quad 0 < |x-a| < \delta \quad$ then $\quad |f(x) - L| < \epsilon$

(b) Show directly from the definition that $\lim_{x\to 3} 2x = 6$

Let $f(x) = 2x$

Part I: Finding δ

1) $|f(x) - 6| = |2x - 6| = |2(x-3)| = 2|x-3|$
2) $2|x-3| < \epsilon$ implies $|x-3| < \frac{\epsilon}{2}$
3) Let $\boxed{\delta = \frac{\epsilon}{2}}$

Part II: Showing your δ works

1) Let $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{2}$, and suppose $0 < |x-3| < \delta$. Then $|x-3| < \frac{\epsilon}{2}$
2) Then $|f(x) - 6| = 2|x-3| < 2\frac{\epsilon}{2} = \epsilon$
3) Hence, if $0 < |x-3| < \delta$, then $|f(x) - 6| < \epsilon$

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2. Let $f(x) = \sqrt{x}$ for all x. Prove directly from the definition of the derivative that $f'(a) = \frac{1}{2\sqrt{a}}$

Solution 1:

$$\begin{aligned}
f'(a) = & \lim_{x\to a} \tfrac{f(x)-f(a)}{x-a} \\
= & \lim_{x\to a} \tfrac{\sqrt{x}-\sqrt{a}}{x-a} \\
= & \lim_{x\to a} \tfrac{\sqrt{x}-\sqrt{a}}{x-a} \times \left(\tfrac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right) \\
= & \lim_{x\to a} \tfrac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)} \\
= & \lim_{x\to a} \tfrac{\left(\sqrt{x}\right)^2-\left(\sqrt{a}\right)^2}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)} \\
= & \lim_{x\to a} \tfrac{x-a}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)} \\
= & \lim_{x\to a} \tfrac{1}{\left(\sqrt{x}+\sqrt{a}\right)} \\
= & \tfrac{1}{\sqrt{a}+\sqrt{a}} \\
= & \boxed{\tfrac{1}{2\sqrt{a}}}
\end{aligned}$$

Solution 2:

$$\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \times \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{a+h}-\sqrt{a}\right)\left(\sqrt{a+h}+\sqrt{a}\right)}{h\left(\sqrt{a+h}+\sqrt{a}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{a+h}\right)^2-\left(\sqrt{a}\right)^2}{h\left(\sqrt{a+h}+\sqrt{a}\right)} \\
&= \lim_{h\to 0} \frac{a+h-a}{h\left(\sqrt{a+h}+\sqrt{a}\right)} \\
&= \lim_{h\to 0} \frac{h}{h\left(\sqrt{a+h}+\sqrt{a}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} \\
&= \frac{1}{\sqrt{a+0}+\sqrt{a}} \\
&= \frac{1}{\sqrt{a}+\sqrt{a}} \\
&= \boxed{\tfrac{1}{2\sqrt{a}}}
\end{aligned}$$

3. Let $f(x) = \sqrt{3 - e^{2x}}$
   (a) Explain why f is one-to-one

   Solution 1:

$$\begin{aligned} f(x) &= f(y) \\ \sqrt{3-e^{2x}} &= \sqrt{3-e^{2y}} \\ 3-e^{2x} &= 3-e^{2y} \\ e^{2x} &= e^{2y} \\ \ln\left(e^{2x}\right) &= \ln\left(e^{2y}\right) \\ 2x &= 2y \\ x &= y \end{aligned}$$

   Hence f is one-to-one.

   Solution 2: e^{2x} is increasing, so $-e^{2x}$ is decreasing, so $3-e^{2x}$ is decreasing, and hence $f(x) = \sqrt{3-e^{2x}}$ is (strictly) decreasing (as a composition of an increasing and a decreasing function). And hence f is one-to-one.

   (b) What is the domain of f^{-1}?
   Solution 1: Using our formula for f^{-1} in question (c), we see that we need $3 - x^2 > 0$ (we want the number under the $\ln$ to be positive), so $x^2 < 3$, so $-\sqrt{3} < x < \sqrt{3}$. **However,** notice that we also want $x \geq 0$ (because $f(x) \geq 0$), so $\boxed{\text{Domain } f^{-1} = [0, \sqrt{3})}$.
   Solution 2:
   First, let's find the domain of f (this will be useful in a second). The only thing we need is that $3 - e^{2x} \geq 0$ (the number under the square root be nonnegative)

$$\begin{aligned} 3-e^{2x} &\geq 0 \\ -e^{2x} &\geq -3 \\ e^{2x} &\leq 3 \\ 2x &\leq \ln(3) \\ x &\leq \frac{\ln(3)}{2} \end{aligned}$$

   So the domain of f is $\left(-\infty, \frac{\ln(3)}{2}\right]$
   Now, because f is decreasing, and $f\left(\frac{\ln(3)}{2}\right) = 0$ and $\lim_{x\to-\infty} f(x) = \sqrt{3}$, we get that the range of f is $[0, \sqrt{3})$. But the range of f is the domain of f^{-1}, so $\boxed{\text{Domain of } f^{-1} = [0, \sqrt{3})}$

(c) Find a formula for f^{-1}

1) Let $y = \sqrt{3 - e^{2x}}$

2)

$$\begin{aligned} y =& \sqrt{3 - e^{2x}} \\ y^2 =& 3 - e^{2x} \\ y^2 - 3 =& -e^{2x} \\ 3 - y^2 =& e^{2x} \\ e^{2x} =& 3 - y^2 \\ 2x =& \ln\left(3 - y^2\right) \\ x =& \frac{\ln\left(3 - y^2\right)}{2} \end{aligned}$$

3) $\boxed{f^{-1}(x) = \frac{\ln\left(3 - x^2\right)}{2}}$

4. A- **Problem 3 on page 127**

(a) $\boxed{-4}$ (f not defined at -4), $\boxed{-2}$ ($\lim_{x\to -2} f(x)$ does not exist), $\boxed{2}$ (ditto), $\boxed{4}$ (ditto)

(b)
- -4 : Neither
- -2: Continuous from the left
- 2: Continuous from the right
- 4: Continuous from the right

B- **Problem 37 on page 164**

- $\boxed{-4}$: Graph has a 'kink' at -4 (left-hand-side limits and right-hand-side limits not equal)
- $\boxed{0}$: Not continuous at 0

C- **Problem 39 on page 164**

- $\boxed{-1}$: Vertical tangent line at -1
- $\boxed{4}$: Graph has a 'kink' at 4

5. Compute:

(a) $\lim_{x\to 3^+} \frac{x^2-9}{x^2+2x-3}$

We have $\lim_{x\to 3^+} x^2-9=0$ by continuity of $y=x^2-9$ while $\lim_{x\to 3^+} x^2+2x-3 = 3^2+2(3)-3 = 9+6-3=12$ by continuity of $y=x^2+2x-3$, and hence:

$$\lim_{x\to 3^+} \frac{x^2-9}{x^2+2x-3} = \frac{0}{12} = \boxed{0}$$

(b)

$$\lim_{x\to 3} \frac{x^2-9}{x^2-2x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(x+1)} = \lim_{x\to 3} \frac{x+3}{x+1} = \frac{3+3}{3+1} = \frac{6}{4} = \boxed{\tfrac{3}{2}}$$

(c)

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2-9}}{2x+5} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(1-\frac{9}{x^2}\right)}}{2x+5} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{1-\frac{9}{x^2}}}{2x+5} \\
&= \lim_{x\to\infty} \frac{|x|\sqrt{1-\frac{9}{x^2}}}{2x+5} \\
&= \lim_{x\to\infty} \frac{x\left(\sqrt{1-\frac{9}{x^2}}\right)}{2x+5} \qquad \text{because } x>0 \\
&= \lim_{x\to\infty} \frac{x\left(\sqrt{1-\frac{9}{x^2}}\right)}{x\left(2+\frac{5}{x}\right)} \\
&= \lim_{x\to\infty} \frac{\sqrt{1-\frac{9}{x^2}}}{\left(2+\frac{5}{x}\right)} \\
&= \frac{\sqrt{1-0}}{2+0} \\
&= \boxed{\tfrac{1}{2}}
\end{aligned}$$

(d) First of all,

$$\frac{\cot(2t)}{t} = \frac{\frac{\cos(2t)}{\sin(2t)}}{t} = \frac{\cos(2t)}{t\left(\sin(2t)\right)}$$

Now $\lim_{t\to 0^+} \cos(2t) = \lim_{t\to 0^+} \cos(2t) = 1$
And $\lim_{t\to 0^+} t\left(\sin(2t)\right) = 0^+$ and $\lim_{t\to 0^-} t\left(\sin(2t)\right) = 0^+$ (in the second case, it's because both t and $\sin(2t)$ are negative!)

Hence:

$$\lim_{t\to 0^+} \frac{\cot(2t)}{t} = \lim_{t\to 0^+} \frac{\cos(2t)}{t\,(\sin(2t))} = \frac{1}{0^+} = \infty$$

and

$$\lim_{t\to 0^-} \frac{\cot(2t)}{t} = \lim_{t\to 0^-} \frac{\cos(2t)}{t\,(\sin(2t))} = \frac{1}{0^+} = \infty$$

And thus:

$$\lim_{t\to 0} \frac{\cot(2t)}{t} = \boxed{\infty}$$

6. Compute:

(a)

$$\frac{d}{dt}\left(t^{\frac{1}{3}}\sec(t)\right) = \boxed{\tfrac{1}{3}t^{-\frac{2}{3}}\sec(t) + t^{\frac{1}{3}}\sec(t)\tan(t)}$$

(b)

$$\frac{d}{du}\left(\frac{u}{u^2+1}\right) = \frac{1\cdot\left(u^2+1\right) - u\cdot(2u)}{\left(u^2+1\right)^2} = \frac{u^2+1-2u^2}{\left(u^2+1\right)^2} = \boxed{\tfrac{1-u^2}{(u^2+1)^2}}$$

7. Find an equation for the line which is normal to the curve consisting of all points (x, y) satisfying $y = \frac{xe^x}{x^2+1}$ at the point $(0, 0)$ on this curve.

The normal line goes through the point $(0, 0)$ and has slope $-\frac{1}{y'(0)}$ (the negative reciprocal of the slope of the tangent line to the graph at 0).

Now:

$$y' = \frac{(xe^x)' \cdot \left(x^2+1\right) - xe^x \cdot (2x)}{\left(x^2+1\right)^2} = \frac{(e^x + xe^x)\left(x^2+1\right) - 2x^2e^x}{\left(x^2+1\right)^2}$$

Hence:

$$y'(0) = \frac{\left(e^0 + 0e^0\right)\left(0^2+1\right) - 2(0)^2e^0}{\left((0)^2+1\right)^2} = \frac{(1)\,(1)}{(1)} = 1$$

And hence the equation of the normal line is:

$$y - 0 = -\frac{1}{1}\,(x - 0)$$

That is: $\boxed{y = -x}$

8. **Problem 50 on page 190**

(a)

$$P'(2) = F'(2)G(2) + F(2)G'(2) = 0 \times 2 + 3 \times \frac{1}{2} = \boxed{\frac{3}{2}}$$

(b)

$$Q'(7) = \frac{F'(7)G(7) - F(7)G'(7)}{G(7)^2} = \frac{\frac{1}{4} \times 1 - 5 \times \left(-\frac{2}{3}\right)}{1^2} = \frac{1}{4} + \frac{10}{3} = \boxed{\frac{43}{12}}$$